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REMARKS ON FORCED EQUATIONS OF THE DOUBLE PENDULUM TYPE

GABRIELLA TARANTELLO

ABSTRACT. Motivated by the double pendulum equation we consider Lagrangian systems with potential $V = V(t, q)$ periodic in each of the variables t, $q = (q_1, \dots, q_N)$. We study periodic solutions for the corresponding equation of motion subject to a periodic force $f = f(t)$. If f has mean value zero, the corresponding variational problem admits a $\mathbf{Z}^N$ symmetry which yields $N+1$ distinct periodic solutions (see [9]). Here we consider the case where the average of f, though bounded, is no longer required to be zero. We show how this situation becomes more delicate, and in general it is only possible to claim no more than two periodic solutions.

INTRODUCTION

Let $q, \xi \in \mathbf{R}^N$ and $t \in \mathbf{R}$. Given the potential $V = V(t, q)$, consider the time-dependent Lagrangian

$$\mathscr{L}(q, \xi, t) = \tfrac{1}{2}A(t, q)\xi \cdot \xi - V(t, q) \tag{0.1}$$

(Here $\cdot$ stands for the usual scalar product in $\mathbf{R}^N$.) where $A = A(t, q)$ is a symmetric, positive definite $N \times N$ matrix.

The equation of motion for the corresponding mechanical system subject to the forcing $f = f(t) \in \mathbf{R}^N$ is given by

$$\frac{d}{dt}\frac{\partial \mathscr{L}}{\partial \xi}(q, \dot{q}, t) - \frac{\partial \mathscr{L}}{\partial q}(q, \dot{q}, t) = f(t). \tag{0.2}$$

Assuming $A = A(t, q)$, $V = V(t, q)$, and $f = f(t)$ time-periodic with same period T, a natural question to ask is whether or not (0.2) admits T-periodic solutions. Obviously the answer depends upon the nature of the potential V.

Motivated by the double pendulum equation we shall consider periodic potentials. More precisely, given $T_k > 0$, $k = 1, \dots, N$, assume

(V) $V = V(t, q) \in C^1$ and $V(t + kT, q + (k_1T_1, \dots, k_NT_N)) = V(t, q)$ $\forall (t, q) \in \mathbf{R} \times \mathbf{R}^N$ and $k, k_s \in \mathbf{Z}$, $s = 1, \dots, N$;

(A) $A = A(t, q) \in C^1$ is symmetric positive definite $N \times N$ matrix and $A(t+kT, q+(k_1T_1, \dots, k_NT_N)) = A(t, q)$ $\forall (t, q) \in \mathbf{R} \times \mathbf{R}^N$, $k, k_s \in \mathbf{Z}$, $s = 1, \dots, N$.

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This situation has attracted the attention of several authors (e.g., [2, 3, 4, 8]) and multiple forced oscillations have been obtained for (0.2) provided the forcing term $f = f(t)$ has mean value zero (i.e., $\int_0^T f(t)\,dt = 0$). Here we investigate the problem without this restriction on the mean value of f. However, our physical intuition suggests that $|\frac{1}{T}\int_0^T f|$ cannot be too large in order for (0.2) to admit periodic solutions.

This can be seen rigorously if, for example, we consider the Lagrangian

$$\begin{aligned}\mathscr{L}(\theta, \phi, \xi_1, \xi_2) = \tfrac{1}{2}((m_1+m_2)l_1^2\xi_1^2 + 2m_2l_1l_2\cos(\theta-\phi)\xi_1\xi_2 + m_2l_2^2\xi_2^2)\\ + g(m_1+m_2)l_1\cos\theta + gm_2l_2\cos\phi\\ (g = \text{constant of gravitation})\end{aligned}$$

corresponding to a coplanar double pendulum with masses m_i and length l_i, $i = 1, 2$. Indeed, if $f(t) = (f_1(t), f_2(t))$, then in this case (0.2) reduces to

$$\begin{aligned}&\frac{d}{dt}((m_1+m_2)l_1^2\dot\theta + m_2l_1l_2\cos(\phi-\theta)\dot\phi)\\ &\qquad - m_2l_1l_2\sin(\phi-\theta)\dot\theta\dot\phi + g(m_1+m_2)l_1\sin\theta = f_1,\\ &\frac{d}{dt}(m_2l_1l_2\cos(\phi-\theta)\dot\theta + m_2l_2^2\dot\phi)\\ &\qquad + m_2l_1l_2\sin(\phi-\theta)\dot\theta\dot\phi + gm_2l_2\sin\phi = f_2.\end{aligned} \tag{0.3}$$

So, summing up the equations in (0.3) and integrating in $[0, T]$ we see that for (0.3) to admit T-periodic solutions, it is necessary that

$$\frac{1}{T}\int_0^T f_1 + \frac{1}{T}\int_0^T f_2 = g((m_1+m_2)l_1\sin\tau_1 + m_2l_2\sin\tau_2)$$

for suitable $\tau_1, \tau_2 \in [0, 2\pi)$. This condition, however, is not sufficient in general. In fact, for example, if we take $\frac{1}{T}\int_0^T f_1 = g(m_1+m_2)l_1$ and $\frac{1}{T}\int_0^1 f_2 = gm_2l_2$ then (0.3) admits a T-periodic solution only if $f_1(t) = g(m_1+m_2)l_1$ and $f_2(t) = gm_2l_2$ $\forall t$. Thus the search of T-periodic solutions for (0.2) becomes a delicate problem, especially when $\int_0^T f \neq 0$. To the author's knowledge the only known results in this direction are those obtained in [9] specifically for the N-pendulum equations.

Here we treat the case where only one of the f components, say the first one, has mean values zero. So we shall write $f(t) = (f_1(t)+c, f_2(t), \dots, f_N(t))$ with $\int_0^T f_k = 0$, $k = 1, \dots, N$ and $c \in \mathbf{R}$.

Notice that under the given assumptions the problem admits a $\mathbf{Z}^N$ symmetry, in the sense that if $q = q(t)$ is a T-periodic solution for (0.2) so it is $q(t) + (k_1T_1, \dots, k_NT_N)$ $\forall k_s \in \mathbf{Z}$, $s = 1, \dots, N$. This motivates the following

Definition. $q_1 = q_1(t)$ and $q_2 = q_2(t)$ are called *distinct* if

$$q_1(t) - q_2(t) \notin \{(k_1T_1, \dots, k_nT_n)\ \forall k_s \in \mathbf{Z},\ s = 1, \dots, N\}$$

for all $t \in [0, T]$.

To clarify the content of our main result (Theorem 1 below) we start with some preliminary observations. First of all we have a variational principle associated with (0.2). Hence T-periodic solutions of (0.2) are critical points of a suitable functional I_c defined in the Hilbert space

$$H = \{q = (q_1, \dots, q_N) : q_k \in H^1([0, T]),\ q_k(0) = q_k(T),\ k = 1, \dots, N\}$$

with $I_c(q) = I_0(q) + c\int_0^T q_1$ for all $q = (q_1, \dots, q_N) \in H$.

Now, if $c = 0$ (i.e., $\int_0^T f = 0$) then I_0 is in fact well defined (and bounded below) in the Hilbert manifold $M = T^N \times H_\#$ where T^N is the N-dimensional torus and $H_\# = \{q \in H : \int_0^T q = 0\}$. So by the Ljusternik-Schnirelman theory one concludes that I_0 admits at least $N+1$ ($=$ cup length $T^N + 1$) distinct critical points (see [7, 8] and [3, 4] for sharper results concerning the N-pendulum equation). This is no longer available when $c \neq 0$, and in fact the behavior of I_c, as far as critical points are concerned, can be completely different from that of I_0 even for very small c. This is illustrated by the following finite-dimensional examples where, in analogy, we investigate the critical points for functions of the type $G_c(x_1, \dots, x_N) = g(x_1, \dots, x_N) + cx_1$ with g periodic in each variable.

Example 1. Take $\psi = \psi(t)$ to be a smooth 1-periodic function. Given $T_k > 0$, $k = 1, \dots, N$, set

$$g(x_1, \dots, x_N) = \psi\left(\sum_{k=1}^N T_k^{-1} x_k\right).$$

So g is T_k periodic in x_k. It is easy to see that while g admits infinitely many critical points, $G_c(x_1, \dots, x_N) = g(x_1, \dots, x_N) + cx_1$ has no critical points for all $c \neq 0$.

Example 2. Take

$$g(x_1, \dots, x_N) = \sin x_1(\sin x_2 - 1)\cdot(\sin x_3 - 1) \cdot \dots \cdot (\sin x_N - 1) + 2^{N-1}\sum_{k=2}^N \sin x_k$$

so g is 2π-periodic in x_k, $k = 1, \dots, N$.

Easy computations show that g has infinitely many critical points in $[0, 2\pi)$. However $G_c(x_1, \dots, x_N) = g(x_1, \dots, x_N) + cx_N$ admits exactly *two* (nondegenerate) critical points in $[0, 2\pi)$ if $|c| < 2^{N-1}$, exactly *one* if $|c| = 2^{N-1}$, and none if $|c| > 2^{N-1}$.

These two examples justify the following.

Theorem 1. *Let $V = V(t, q)$ and $A = A(t, q)$ satisfy (V) and (A), respectively, and $f_k = f_k(t)$ be a T-periodic continuous function with $\int_0^T f_k = 0$,*

$k = 1, \dots, N$. There exist constants $d \le 0 \le D$ (depending on V, A, and f_k) with the following property:

(i) *if $d = D$ then for all $\xi \in \mathbf{R}$ problem (0.2) with $f(t) = (f_1(t), \dots, f_N(t))$ admits a T-periodic solution $q_\xi(t) = (q_{1,\xi}(t), \dots, q_{N,\xi}(t))$ with $\frac{1}{T}\int_0^T q_{1,\xi} = \xi$;*

(ii) *if $d < D$ then problem (0.2) with $f(t) = (f_1(t) + c, \dots, f_N(t))$ admits at least two distinct solutions if $c \in (d, D)$, and at least one solution if $c = d$ or $c = D$.* □

Remark. Restrictions on c were expected by the previous observations. Furthermore, by virtue of Examples 1 and 2 we know that in general the statement cannot be improved. However, we shall see how, in certain cases, it is possible to rule out the possibility $d = 0 = D$ and give estimates on d and D. This will be the content of Theorem 2 below. Here we only state its particular implication to the double pendulum equation.

Corollary. *Let $e(t) = (e_1(t)), e_2(t))$ be a continuous T-periodic function with $\int_0^T e_k = 0$, $k = 1, 2$, and $c \in \mathbf{R}$. Assume*

(a) $(m_1 + m_2)l_1 - m_2 l_2 := \gamma > 0$ *and*

(b) $((m_1 + m_2)l_1/2\lambda_0\pi^2)(\|e\|_2 + \sqrt{T}(m_1 + m_2)l_1)T^{3/2} := \sigma_T < \gamma$ *with*

$$\lambda_0 = \tfrac{1}{2}(m_2 l_2^2 + (m_1 + m_2)l_1^2 - \sqrt{(m_2 l_2^2 - (m_1 + m_2)l_1^2)^2 + 4m_2^2 l_1^2 l_2^2}). \tag{0.4}$$

There exist constants $d < 0 < D$ (depending on e_k, m_k, l_k, $k = 1, 2$, and T) such that equation (0.3) with $f_1(t) = e_1(t) + c$ and $f_2(t) = e_2(t)$ admits at least two distinct solutions if $c \in (d, D)$ and at least one solution if $c = d$ or $c = D$. Furthermore, $d \le -\gamma + \sigma_T < 0 < \gamma - \sigma_T \le D$. □

Similar conditions were introduced in [4] to obtain multiple forced oscillations for the double pendulum equation in case $c = 0$. See [3], also, for extensions to the N-pendulum case. Furthermore, it has been observed in [9] that pathologies of the kind portrayed by Examples 1 and 2 cannot occur for the N-pendulum equation, where much stronger multiplicity results hold provided m_i, l_1, $e_i(t)$, and T satisfy suitable conditions. For the simple pendulum equation, a result of the type of Theorem 1 was obtained by Mawhin-Willem [7] (see also [11]).

The proof of Theorem 1 *and applications.* Set $f(t) = (f_1(t)+c, \dots, f_N(t))$ with $f_k \in L^2[0, T]$, $\int_0^T f_k = 0$, $k = 1, \dots, N$, and $c \in \mathbf{R}$. We seek solutions for the following problem

$$(1)_c \qquad \begin{cases} \dfrac{d}{dt}\dfrac{\partial \mathscr{L}}{\partial \xi_1}(q, \dot q, t) - \dfrac{\partial \mathscr{L}}{\partial q_1}(q, \dot q, t) = f_1(t) + c, \\ \dfrac{d}{dt}\dfrac{\partial \mathscr{L}}{\partial \xi_k}(q, \dot q, t) - \dfrac{\partial \mathscr{L}}{\partial q_k}(q, \dot q, t) = f_k(t), \quad k = 2, \dots, N, \\ q(0) = q(T), \qquad \dot q(0) = \dot q(T). \end{cases}$$

As it is well known from classical mechanics, (weak) solutions of $(1)_c$ are the critical points of the functional

$$I_c(q) = \frac{1}{2}\int_0^T A(t, q)\dot{q}\cdot\dot{q} - \int_0^T V(t, q) + \sum_{k=1}^N \int_0^T f_k q_k + c\int_0^T q_1$$

defined in the Hilbert space

$$H = \{q = (q_1, \dots, q_N) : q_k \in H^1([0, T]),\ q_k(0) = q_k(T),\ k = 1, \dots, N\}$$

equipped with the standard scalar product

$$(q, Q) = \int_0^T \dot{q}\cdot\dot{Q} + \int_0^T q\cdot Q, \qquad q, Q \in H,$$

and norm $\|q\| = (q, q)^{1/2}$. Notice that for $c \neq 0$, I_c is unbounded in H and multivalued in $M = T^N \times H_\#$.

For $q = (q_1, \dots, q_N) \in H$, set

$$\|q\|_2 = \left(\sum_{k=1}^N \|q_k\|_{L^2}^2\right)^{1/2} \quad \text{and} \quad \|q\|_\infty = \sum_{k=1}^N \|q_k\|_{L^\infty}.$$

Moreover, since $A = A(t, q)$ is positive definite, let $\lambda_0 > 0$ satisfy $A(t, q)\xi\cdot\xi \geq \lambda_0|\xi|^2$ $\forall t \in \mathbf{R}$, $q, \xi \in \mathbf{R}^N$. For every $c \in \mathbf{R}$, we have that I_c satisfies a Palais-Smale type condition. More precisely,

Lemma 1.1. *Any sequence $q_n \in H$ satisfying:*

(i) $|\frac{1}{T}\int_0^T q_n| \leq C_1$ *for all* $n \in \mathbf{N}$ (C_1 *positive constant*),
(ii) $I_c(q_n) \leq C_2$ *for all* $n \in \mathbf{N}$ (C_2 *positive constant*),
(iii) $(I_c'(q_n) - I_c'(q_m))(q_n - q_m) \to 0$ *as* $n, m \to +\infty$, *admits a convergent subsequence in* H.

Proof. Since the potential V is bounded, set $|V(t, q)| \leq V_0$ $\forall (t, q) \in \mathbf{R}\times\mathbf{R}^N$ for a suitable constant $V_0 > 0$. Furthermore, let $f_0(t) = (f_1(t), \dots, f_N(t))$ and $q_n = q_n^0 + \xi_n$ with $\xi_n = (\xi_{1,n}, \dots, \xi_{N,n}) \in \mathbf{R}^N$ and $\int_0^T q_n^0 = 0$. We have

$$\begin{aligned} C_2 \geq I_c(q_n) &= \frac{1}{2}\int_0^T A(t, q_n)\dot{q}_n\cdot\dot{q}_n - \int_0^T V(t, q_n) \\ &\quad + \int_0^T f_0\cdot q_n^0 + c\int_0^T q_1 \\ &\geq \frac{\lambda_0}{2}\|\dot{q}_n\|_2^2 - \|f_0\|_2\|q_n^0\|_2 - (V_0 + |c|C_1)T \\ &\geq \frac{\lambda_0}{2}\|\dot{q}_n\|^2 - \frac{T\|f_0\|_2}{2\pi}\|\dot{q}_n\|_2 - (V_0 + |c|C_1)T. \end{aligned}$$

This yields $\|\dot{q}_n\|_2 \leq c_1$ $\forall n \in \mathbf{N}$ for some suitable constant $c_1 > 0$. This fact, together with (i) implies that for a subsequence of q_n (which we still call q_n) and $q \in H$ we have $\|q_n - q\|_\infty \to 0$ as $n \to +\infty$. Set

$$A_n(t) = \left(\frac{\partial A}{\partial q_1}(t, q_n)\dot{q}_n\cdot\dot{q}_n, \dots, \frac{\partial A}{\partial q_N}(t, q_n)\dot{q}_n\cdot\dot{q}_n\right) \in \mathbf{R}^N.$$

Obviously,

$$\sum_{k=1}^{N}\left\|\frac{\partial A}{\partial q_k}(t, q_n)\dot{q}_n\cdot\dot{q}_n\right\|_{L^1}\le c_2\quad\forall n\in\mathbf{N}$$

for $c_2>0$ suitable constant. Furthermore, by the Lipschitz continuity of $A=A(t, q)$ in q (uniformly in t) we have

$$\begin{aligned}(I_c'(q_n)-I_c'(q_m))(q_n-q_m)&=\int_0^T A(t, q_n)(\dot{q}_n-\dot{q}_m)\cdot(\dot{q}_n-\dot{q}_m)\\&+\int_0^T (A(t, q_n)-A(t, q_m))\dot{q}_m\cdot(\dot{q}_n-\dot{q}_m)\\&+\frac{1}{2}\int_0^T (A_n(t)-A_m(t))\cdot(q_n-q_m)\\&+\int_0^T\left(\frac{\partial V}{\partial q}(t, q_n)-\frac{\partial V}{\partial q}(t, q_m)\right)\cdot(q_n-q_m)\\&\ge\lambda_0\|\dot{q}_n-\dot{q}_m\|_2^2-c_3\|q_n-q_m\|_\infty\quad(c_3\text{ positive constant}).\end{aligned}$$

Consequently, $\|\dot{q}_n-\dot{q}_m\|_2\to 0$ as $n, m\to+\infty$, and therefore $q_n\to q$ in H. □

Given $\xi\in\mathbf{R}$ set $\Lambda_\xi=\{q=(q_1, \dots, q_N)\in H:\frac{1}{T}\int_0^T q_1=\xi\}$. For every $c\in\mathbf{R}$, I_c is bounded from below in Λ_ξ. Furthermore,

Lemma 1.2. *For every $\xi\in\mathbf{R}$, there exists $q_\xi\in\Lambda_\xi$ such that*

$$I_0(q_\xi)=\inf_{\Lambda_\xi} I_0. \tag{1.1$_\xi$}$$

Moreover, $\operatorname{Inf}_{\Lambda_\xi} I_c=I_0(q_\xi)+cT\xi=I_c(q_\xi)$.

Proof. Set $m_\xi=\inf_{\Lambda_\xi} I_0$ and let $q_n=(q_{1,n}, \dots, q_{N,n})\in\Lambda_\xi$ such that

$$\lim_{n\to+\infty} I_0(q_n)=m_\xi. \tag{1.2}$$

Since $I_0(q)=I_0(q+(k_1T_1, \dots, k_nT_n))$ for all $k_s\in\mathbf{Z}$, $s=1, \dots, N$, we can always assume that $\frac{1}{T}\int_0^T q_{k,n}\in[0, T_k]$, $k=2, \dots, N$. So

$$\left|\frac{1}{T}\int_0^T q_n\right|\le|\xi|+\sum_{k=2}^{N}T_k. \tag{1.3}$$

Furthermore from (1.2) it follows that $\|\dot{q}_n\|_{L^2}\le C$ $\forall n\in\mathbf{N}$, for a suitable positive constant C. Hence for a subsequence $\{q_{n_k}\}$ of $\{q_k\}$ and $q_0\in H$ we have $\dot{q}_{n_k}\to\dot{q}_0$ weakly in L^2 and $q_{n_k}\to q_0$ strongly in L^2.

Consequently $q_0\in\Lambda_\xi$. The lower semicontinuity of I_0 yields the conclusion. □

Our next goal is to obtain a priori estimates on $\dot{q}_\xi$ independent of ξ. To this purpose let L be the Lipschitz constant of V, i.e.,

$$|V(t, q)-V(t, Q)|\le L|q-Q| \tag{1.4}$$

for all $t\in\mathbf{R}$, $q, Q\in\mathbf{R}^N$. Set $f_0(t)=(f_1(t), \dots, f_N(t))$. We have

Lemma 1.3. *If* $q_\xi \in \Lambda_\xi$ *satisfies* $(1.1)_\xi$ *then*

$$\|\dot q_\xi\|_2 \le \frac{T}{\pi\lambda_0}(\|f_0\|_2 + \sqrt{T}L).$$

Proof. Set $q_\xi = q_\xi^0 + (\xi, \sigma_2, \dots, \sigma_N)$ with $\int_0^T q_\xi^0 = 0$. Since $TV(\xi, \sigma_2, \dots, \sigma_N) = I_0(\xi, \sigma_2, \dots, \sigma_N) \ge I_0(q_\xi)$, we have

$$\begin{aligned}0 &\ge \frac{1}{2}\int_0^T A(t, q_\xi)\dot q_\xi \cdot \dot q_\xi \\ &\quad - \int_0^T (V(q_\xi^0 + (\xi, \sigma_2, \dots, \sigma_N)) - V(\xi, \sigma_2, \dots, \sigma_N)) + \int_0^T f_0 \cdot q_\xi^0 \\ &\ge \frac{\lambda_0}{2}\|\dot q_\xi\|_2^2 - L\sqrt{T}\|q_\xi^0\|_2 - \|f_0\|_2\|q_\xi^0\|_2.\end{aligned}$$

Thus

$$0 \ge \|\dot q_\xi\|_2\left(\frac{\lambda_0}{2}\|\dot q_\xi\|_2 - \frac{T}{2\pi}(\|f_0\|_2 + L\sqrt{T})\right). \quad \square$$

For $\xi \in \mathbf{R}$ set $\Gamma_\xi = \{q \in \Lambda_\xi : I_0(q) = \inf_{\Lambda_\xi} I_0\}$. By Lemma 1.2 we know that $\Gamma_\xi \ne \varnothing$ for all $\xi \in \mathbf{R}$.

Lemma 1.4. *For every* $c \in \mathbf{R}$, *there exists* $L_c > 0$ *such that*

$$|I_c(q_{\xi_1} + \sigma_1) - I_c(q_{\xi_2} + \sigma_2)| \le L_c(\|q_{\xi_1} - q_{\xi_2}\| + |\sigma_1 - \sigma_2|)$$

for all $\sigma_i \in \mathbf{R}^N$, $\xi_i \in \mathbf{R}$, $q_{\xi_i} \in \Gamma_{\xi_i}$, $i = 1, 2$.

Proof. Let L' be the (uniform) Lipshitz constant corresponding to $A(t, q)$ and let A_0 satisfy $|A(t, q)\xi \cdot \eta| \le A_0|\xi|\,|\eta|$ for all $t \in \mathbf{R}$, $q, \xi, \eta \in \mathbf{R}^N$. Set $\hat q_{\xi_i} = q_{\xi_i} + \sigma_i$, $i = 1, 2$. We have

$$\begin{aligned}&|I_c(\hat q_{\xi_1}) - I_c(\hat q_{\xi_2})| \\ &\quad\le \frac{1}{2}\int_0^T |A(t, \hat q_{\xi_1})\dot q_{\xi_1} \cdot \dot q_{\xi_1} - A(t, \hat q_{\xi_2})\dot q_{\xi_2} \cdot \dot q_{\xi_2}| \\ &\qquad + \int_0^T |V(t, \hat q_{\xi_1}) - V(t, \hat q_{\xi_2})| + \int_0^T |f_0 \cdot (q_{\xi_1}^0 - q_{\xi_2}^0)| \\ &\qquad + T|c|(|\xi_1 - \xi_2| + |\sigma_1 - \sigma_2|) \\ &\quad\le \frac{1}{2}\int_0^T |(A(t, \hat q_{\xi_1}) - A(t, \hat q_{\xi_2}))\dot q_{\xi_1} \cdot \dot q_{\xi_1}| + \frac{1}{2}\int_0^T |A(t, \hat q_{\xi_2})\dot q_{\xi_1} \cdot (\dot q_{\xi_1} - \dot q_{\xi_2})| \\ &\qquad + \frac{1}{2}\int_0^T |A(t, \hat q_{\xi_2})\dot q_{\xi_2} \cdot (\dot q_{\xi_1} - \dot q_{\xi_2})| + L\sqrt{T}\|\hat q_{\xi_1} - \hat q_{\xi_2}\|_2 \\ &\qquad + \frac{T}{2\pi}\|f_0\|_2\|\dot q_{\xi_1} - \dot q_{\xi_2}\|_2 + |c|T(|\xi_1 - \xi_2| + |\sigma_1 - \sigma_2|) \\ &\quad\le \frac{L'}{2}\|\hat q_{\xi_1} - \hat q_{\xi_2}\|_\infty\|\dot q_{\xi_1}\|_2^2 + \frac{A_0}{2}\|\dot q_{\xi_1}\|_2\|\dot q_{\xi_1} - \dot q_{\xi_2}\|_2 + \frac{A_0}{2}\|\dot q_{\xi_2}\|_2\|\dot q_{\xi_1} - \dot q_{\xi_2}\|_2 \\ &\qquad + \sqrt{T}(L + |c|)\|q_{\xi_1} - q_{\xi_2}\|_2 + \frac{T}{2\pi}\|f_0\|_2\|\dot q_{\xi_1} - \dot q_{\xi_2}\|_2 + T(|c| + L)|\sigma_1 - \sigma_2|.\end{aligned}$$

Hence by the estimate of Lemma 1.3 we conclude that

$$\begin{aligned}|I_c(q_{\xi_1}) - I_c(q_{\xi_2})| &\le \frac{L'}{2}\left(\frac{T}{\pi\lambda_0}(\|f_0\|_2 + \sqrt{T}L)\right)^2 (\|q_{\xi_1} - q_{\xi_2}\|_\infty + |\sigma_1 - \sigma_2|) \\ &\quad + \left(A_0\frac{T}{\pi\lambda_0}(\|f_0\|_2 + \sqrt{T}L) + \frac{T}{2\pi}\|f_0\|_2\right)\|\dot q_{\xi_1} - \dot q_{\xi_2}\|_2 \\ &\quad + \sqrt{T}(L + |c|)(\|q_{\xi_1} - q_{\xi_2}\|_2 + \sqrt{T}|\sigma_1 - \sigma_2|) \\ &\le L_c(\|q_{\xi_1} - q_{\xi_2}\| + |\sigma_1 - \sigma_2|)\end{aligned}$$

for some suitable constant $L_c > 0$ independent of ξ_i and σ_i, $i = 1, 2$. □

Proof of Theorem 1. Given $q \in H$, define

$$\psi(q) = -\frac{1}{T}\left(\frac{1}{2}\int_0^T \frac{\partial A}{\partial q_1}(t, q)\dot q \cdot \dot q - \int_0^T \frac{\partial V}{\partial q_1}(t, q)\right).$$

Set $d = \inf_{\xi\in\mathbf{R}} \inf_{q\in\Gamma_\xi} \psi(q)$ and $D = \sup_{\xi\in\mathbf{R}} \sup_{q\in\Gamma_\xi} \psi(q)$. By Lemma 1.3 we know that $-\infty < d \le D < +\infty$. Furthermore, I_0 is bounded from below in H, and by Lemma 1.1 it follows that its minimum is achieved at some point $q_0 \in H$. In particular $\psi(q_0) = 0$. So $d \le \psi(q_0) = 0 \le D$. Notice that $\forall \xi \in \mathbf{R}$ and $q_\xi \in \Gamma_\xi$ we have $I_0'(q_\xi) = -T(\psi(q_\xi), 0, \dots, 0) \in \mathbf{R}^N$. Therefore if $d = D = 0$, then for every $\xi \in \mathbf{R}$, $q_\xi \in \Gamma_\xi$ $(\ne \emptyset)$ would be a solution for $(1)_{c=0}$ and $\frac{1}{T}\int_0^T q_{1,\xi} = \xi$. Now assume $d < D$, and let $c \in (d, D)$. There exists $\xi_1, \xi_2 \in \mathbf{R}$ such that $\psi(q_{\xi_2}) < c < \psi(q_{\xi_1})$ for all $q_{\xi_1} \in \Gamma_{\xi_1}$ and for all $q_{\xi_2} \in \Gamma_{\xi_2}$. Since $\psi(q_{\xi+kT_1}) = \psi(q_\xi)$ and $q_\xi \in \Gamma_\xi \Leftrightarrow q_{\xi+kT_1} \in \Gamma_{\xi+kT_1}$ for all $k \in \mathbf{Z}$, we can always assume $0 < \xi_2 - \xi_1 < T_1$. Set

$$\Lambda_{\xi_1,\xi_2} = \left\{q = (q_1, \dots, q_N) \in H : \xi_1 \le \frac{1}{T}\int_0^T q_1 \le \xi_2\right\}.$$

Notice that I_c is bounded from below in Λ_{ξ_1,ξ_2}. We shall obtain our first solution by showing that $\inf_{\Lambda_{\xi_1,\xi_2}} I_c$ is achieved at an interior point of Λ_{ξ_1,ξ_2}. Set $m = \inf_{\Lambda_{\xi_1,\xi_2}} I_c$. Let $q_n = (q_{1,n}, \dots, q_{N,n}) \in \Lambda_{\xi_1,\xi_2}$ $n \in \mathbf{N}$, satisfy $\lim_{n\to+\infty} I_c(q_n) = m$. If $\xi_n = \frac{1}{T}\int_0^T q_{1,n} \in [\xi_1, \xi_2]$, without loss of generality we can assume $q_n = q_{\xi_n} \in \Gamma_{\xi_n}$ and $\lim_{n\to+\infty}\xi_n = \xi_0$ with $\xi_1 \le \xi_0 \le \xi_2$. In addition, given $q_{\xi_0} \in \Gamma_{\xi_0}$, by Lemma 1.4 we have

$$\begin{aligned} m = \inf_{\Lambda_{\xi_1,\xi_2}} I_c &\le I_c(q_{\xi_0}) \le I_c(q_{\xi_n} + (\xi_0 - \xi_n, 0, \dots, 0)) \\ &= I_c(q_{\xi_n} + (\xi_0 - \xi_n, 0, \dots, 0)) - I_c(q_{\xi_n}) + I_c(q_{\xi_n}) \\ &\le L_c|\xi_0 - \xi_n| + I_c(q_{\xi_n}) \to m \quad \text{as } n \to +\infty.\end{aligned}$$

Therefore $I_c(q_{\xi_0}) = m$. We are done if we show $\xi_1 < \xi_0 < \xi_2$. To this purpose set $\mathscr{T}_1(s) = I_c(q_{\xi_1} + (s, 0, \dots, 0))$. We have

$$\mathscr{T}_1'(0) = \frac{1}{2}\int_0^T \frac{\partial A}{\partial q_1}(t, q_{\xi_1})\dot{q}_{\xi_1} \cdot \dot{q}_{\xi_1} - \int_0^T \frac{\partial V}{\partial q_1}(t, q_{\xi_1}) + cT$$
$$= -T(\psi(q_{\xi_1}) - c) < 0.$$

Similarly, if $\varphi_2(s) = I_c(q_{\xi_2} + (s, 0, \dots, 0))$ then $\varphi_2'(0) = -T(\psi(q_{\xi_2}) - c) > 0$. So for small $\varepsilon > 0$, $I_c(q_{\xi_0}) \le I_c(q_{\xi_1} + (\varepsilon, 0, \dots, 0)) < I_c(q_{\xi_1})$ and $I_c(q_{\xi_0}) \le I_c(q_{\xi_2} - (\varepsilon, 0, \dots, 0)) < I_c(q_{\xi_2})$. Consequently $\xi_0 \ne \xi_1$ and $\xi_0 \ne \xi_2$, which yields $I_c'(q_{\xi_0}) = 0$. So q_{ξ_0} is a solution for $(1)_c$. Moreover, q_{ξ_0} is a local minimum for I_c, so we shall obtain a second solution via a mountain-pass theorem (see [1]). To this purpose notice that $I_c(q) = I_c(q + (0, k_2T_2, \dots, k_NT_N))$ for all $k_s \in \mathbf{Z}$, $s = 2, \dots, N$.

Consider

$$\Sigma = \{\gamma : [0; 1] \to H \text{ continuous} : \gamma(0) = q_{\xi_0},$$
$$\gamma(1) = q_{\xi_0} + (0, k_2T_2, \dots, k_NT_N);\ k_s \in \mathbf{Z}\}$$

and set $\alpha = \inf_{\gamma\in\Sigma} \sup_{t\in[0,1]} I_c(\gamma(t)) \ge I_c(q_{\xi_0})$. Although I_c does not satisfy the Palais-Smale condition in the usual sense, Lemma 1.1 is enough to guarantee a "deformation lemma". It can be obtained by adjusting the standard proof of the deformation theorem as shown in [8, Proposition 1.10], with the further simplification that condition 2^0 of Proposition 1.10 is not needed here. Thus a sharper version of the mountain-pass theorem due to D. Guo, J. Sun, and G. Qi (see [5, Proposition 2]) gives that α is a critical value for I_c, and that $K_\alpha := \{q \in H : I_c(q) = \alpha,\ I_c'(q) = 0\}$ contains a critical point different from $q_{\xi_0} + (k_1T_1, k_2T_2, \dots, k_NT_N)$ for all $k_s \in \mathbf{Z}$, $s = 1, \dots, N$. (This generalizes previous results of Hofer [6].) So we are guaranteed a second *distinct* solution for $(1)_c$.

Finally, if $c = d$ or $c = D$, take a sequence $\{c_n\} \subset (d, D)$ such that $c_n \to d$ (or $c_n \to D$) as $n \to +\infty$. By previous arguments, for every $n \in \mathbf{N}$ there exists $q_n = (q_{1,n}, \dots, q_{N,n}) \in H$ such that

(a) $\frac{1}{T}\int_0^T q_{k,n} \in [0, T_k]$ for all $n \in \mathbf{N}$ and for all $k = 1, \dots, N$;
(b) $\|\dot{q}_n\|_2 \le \frac{T}{\lambda_0\pi}(\|f_0\|_2 + \sqrt{T}L)$;
(c) $0 = I_{c_n}'(q_n) = I_d'(q_n) + c_n - d$.

Therefore $|(I_d'(q_n) - I_d'(q_m))(q_n - q_m)| = |c_n - c_m|\,\|q_n - q_m\| \to 0$ as $n, m \to +\infty$, since (a) and (b) $\Rightarrow \|q_n - q_m\| \le C$ for all $n, m \in \mathbf{N}$ ($C > 0$ constant). Hence as for Lemma 1.1 we obtain a subsequence $\{q_{n_k}\}$ of $\{q_n\}$ and $q \in H : q_{n_k} \to q$ in H. Obviously $I_d'(q) = 0$. Completely analogous is the case $c_n \to D$. □

Our next goal is to find conditions on the matrix $A = A(t, q)$ and $V = V(t, q)$ which exclude the possibility $d = 0 = D$. To this end let $V = V(t, q)$

have the special form

$$V(t, q) = g(q_1) + V_1(t, q) \tag{1.5}$$

where

$(V)_1$ $\qquad g \in C^1$ is T_1 periodic and V_1 satisfies (V).

Set $g'_m = \min_{\mathbf{R}} g' \le 0 \le \max_{\mathbf{R}} g' = g'_M$,

$$V_m = \min_{\mathbf{R}^{N+1}} \sum_{k=1}^{N} \frac{\partial V_1}{\partial q_k} \le \max_{\mathbf{R}^{N+1}} \sum_{k=1}^{N} \frac{\partial V_1}{\partial q_k} = V_M$$

and $\gamma = \min\{-g'_m - V_M, g'_M + V_m\}$. Furthermore, let $\lambda_1 \ge 0$ satisfy

$$\left|\sum_{k=1}^{N} \frac{\partial A}{\partial q_k}(t, q)\xi \cdot \xi\right| \le \lambda_1 |\xi|^2 .$$

We have

Theorem 2. *Let $A = A(t, q)$, $f_k = f_k(t)$, $k = 1, \dots, N$, as in Theorem* 1, *and $V = V(t, q)$ of the form* (1.5) *satisfy* $(V)_1$. *Assume $\gamma > 0$ and*

$$\mu_T := \frac{T}{\lambda_0 \pi^2}(\|f_0\|_2 + \sqrt{T}L)\left(\frac{\lambda_1}{\lambda_0}(\|f_0\|_2 + \sqrt{T}L) + \frac{l}{2}\sqrt{T}\right) < \gamma \tag{1.6}$$

where L and l are the Lipshitz constants of V and g' respectively. There exists constants $d < 0 < D$ such that if $c \in (d, D)$ then problem $(1)_c$ admits at least two distinct solutions; and if $c = d$ or $c = D$ then $(1)_c$ admits at least one solution. Furthermore; $d \le g'_m + V_M + \mu_T < 0 < g'_M + V_m - \mu_T$.

Proof. First of all, notice that for all $\xi \in \mathbf{R}$ and $q_\xi \in \Gamma_\xi$, $\int_0^T \frac{\partial A}{\partial q_k}(t, q_\xi)\dot{q}_\xi \cdot \dot{q}_\xi - \frac{\partial V}{\partial q_k}(t, q_\xi) = 0$ for all $k = 2, \dots, N$. Let $\xi_0 \in [0, T_1]$ with $g'(\xi_0) = g'_m$. We have

$$\begin{aligned} -T\psi(q_{\xi_0}) &= \int_0^T \left(\frac{\partial A}{\partial q_1}(t, q_{\xi_0})\dot{q}_{\xi_0} \cdot \dot{q}_{\xi_0} - g'(q_{1,\xi_0}) - \frac{\partial V_1}{\partial q_1}(t, q_{\xi_0})\right) \\ &= \int_0^T \sum_{k=1}^{N} \frac{\partial A}{\partial q_k}(t, q_{\xi_0})\dot{q}_{\xi_0} \cdot \dot{q}_{\xi_0} - \int_0^T (g'(q^0_{1,\xi_0} + \xi_0) - g'(\xi_0)) \\ &\quad - Tg'_m - \int_0^T \sum_{k=1}^{N} \frac{\partial V_1}{\partial q_k}(t, q_{\xi_0}) \\ &\ge - Tg'_m - \left[TV_M + \lambda_1 \int_0^T |\dot{q}_{\xi_0}|^2 + \frac{lT^{3/2}}{2\pi}\|\dot{q}_{1,\xi_0}\|_2\right] . \end{aligned}$$

So, by the estimate of Lemma 1.3 we conclude

$$
\begin{aligned}
-T\psi(q_{\xi_0}) &\geq -T\left[g'_m + V_m + \frac{\lambda_1 T}{(\lambda_0\pi)^2}(\|f_0\|_2 + \sqrt{T}L)^2 + \frac{lT^{3/2}}{2\lambda_0\pi^2}(\|f_0\|_2 + \sqrt{T}L)\right] \\
&\geq -T\left[g'_m + V_M + \frac{T}{\lambda_0\pi^2}(\|f_0\|_2 + \sqrt{T}L)\left(\frac{\lambda_1}{\lambda_0}(\|f_0\|_2 + \sqrt{T}L) + \frac{l}{2}\sqrt{T}\right)\right] \\
&> 0.
\end{aligned}
$$

That is, $d \leq \psi(q_{\xi_0}) \leq g'_m + V_M + \mu_T < 0$. Similarly, if we let $\xi_1 \in [0, T_1]$ such that $g'(\xi_1) = g'_M$, we have

$$
\begin{aligned}
T\psi(q_{\xi_1}) = &- \int_0^T \sum_{k=1}^N \frac{\partial A}{\partial q_k}(t, q_{\xi_1})\dot{q}_{\xi_1} \cdot \dot{q}_{\xi_1} + \int_0^T (g'(q^0_{1,\xi_1} + \xi_1) = g'(\xi_1)) \\
&+ g'_M T + \int_0^T \sum_{k=1}^N \frac{\partial V_1}{\partial q_k}(t, q_{\xi_1}) \\
&\geq g'_M T + V_m T - \frac{T^2}{\lambda_0\pi^2}(\|f_0\|_2 + \sqrt{T}L)\left(\frac{\lambda_1}{\lambda_0}(\|f_0\|_2 + \sqrt{T}L) + \sqrt{T}\frac{l}{2}\right) > 0.
\end{aligned}
$$

This yields $D \geq g'_M + V_m - \mu_T > 0$. The conclusion now easily follows from Theorem 1. □

APPLICATION TO THE N-PENDULUM EQUATION

Given $m_i > 0$, $l_i > 0$, $i = 1, \dots, N$, set $M_i = \sum_{j=i}^N m_j$. Theorem 2 applies to the Lagrangian

$$
\mathscr{L}(q, \xi) = \frac{1}{2}A(q)\xi \cdot \xi + g\sum_{k=1}^N l_k M_k \cos q_k \tag{1.7}
$$

where $A(q) = \{a_{i,j}(q)\}_{i,j=1,\dots,N}$ and

$$
a_{i,j}(q) = a_{i,j}(q_1, \dots, q_N) = M_{\max(i,j)} l_i l_j \cos(q_i - q_j) \tag{1.8}
$$

which corresponds to the mechanical system of N-coplanar penduli with masses m_i and length l_i, $i = 1, \dots, N$. We obtain

Corollary 1. *Let* $f_0(t) = (f_1(t), \dots, f_N(t)) \in \mathbf{R}^N$ *as in Theorem* 1. *Assume*

$$
M_1 l_1 - \sum_{k=2}^N M_k l_k = \gamma > 0 \tag{1.9}
$$

and

$$
\frac{T^{3/2} M_1 l_1}{2\lambda_0\pi^2}(\|f_0\|_2 + \sqrt{T} M_1 l_1) < \gamma \tag{1.10}
$$

where λ_0 *is the elliptic constant of the matrix* $A = A(q)$ *given in* (1.8). *There exist constants* $d < 0 < D$ *(depending on* m_i, l_i, f_i, $i = 1, \dots, N$ *and* T*) such* *that problem* $(1)_c$ *with* $\mathscr{L}$ *given in* (1.7) *admits at least two distinct solutions if* $c \in (d, D)$ *and at least one solution if* $c = d$ *or* $c = D$. *Furthermore,*

$$d \leq -\gamma + \frac{M_1 l_1 T^{3/2}}{2\lambda_0 \pi^2}(\|f_0\|_2 + \sqrt{T} M_1 l_1) < 0$$

$$< \gamma - \frac{M_1 l_1 T^{3/2}}{2\lambda_0 \pi^2}(\|f_0\|_2 + \sqrt{T} M_1 l_1) \leq D.$$

Proof. Just notice that in this case $L = l = M_1 l_1$ and $\sum_{k=1}^N \frac{\partial A(q)}{\partial q_k} \xi \cdot \xi = 0$ for all $q, \xi \in \mathbf{R}^N$; so we can take $\lambda_1 = 0$. □

Finally, concerning the corollary as stated in the Introduction, notice that it is exactly Corollary 1 with $N = 2$, where λ_0 has been explicitly computed and given by (0.4).

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Department of Mathematics, University of California, Berkeley, California 94720

Current address: Department of Mathematics, Carnegie Mellon University, Pittsburgh, Pennsylvania 15213